Mastermath	Commutative algebra exam	13:30-16:30 28/1/2026

- Write your name, university, and student number on every sheet you hand in.
- You may not use any books or notes, or a calculator, during the exam.
- Unless stated otherwise, you need to give full proofs in all your answers. You are allowed to use results that are treated in the book and lectures.
- If you cannot do a part of a question, you may still use its conclusion later on.
- There are **5** questions in total. The exam continues on the back of this sheet.

(1) At the top of the following table, two ring homomorphisms $f\colon R \to A$ are listed.

	$f\colon \mathbb{Z} \to \mathbb{Q}[x]$	$f\colon \mathbb{Z} \to \frac{\mathbb{Z}}{14\mathbb{Z}}[x]$
A is finitely generated as R-algebra		
A is finitely generated as R-module		
A is flat as R-module	(b)	(c)

(a) **Copy the above table onto your answer paper**, then fill in each box in the table with T or F, according to whether or not the given property is true for the given ring homomorphism f in that column. **You do not need to justify your answers to this part.**

(b) Prove your answer in the box marked (b).

(c) Prove your answer in the box marked (c).

(2) Let k be a field. Let $I = (xy, xz, yz^2, z^3) \subseteq k[x, y, z]$.

(a) Compute the radical $\sqrt{I}$ of I.

(b) Compute a minimal primary decomposition of $\sqrt{I}$. Justify your answer.

(c) Compute a minimal primary decomposition of I. Justify your answer.

(d) List the embedded associated primes of I.

(e) For each associated prime $\mathfrak{p}$ of I, find an element f in $k[x, y, z]$ such that $\mathfrak{p}$ equals the radical of $(I : f)$. (Recall that $(I : f)$ is convenient notation for $(I : (f))$.)

(3) Let k be a field with characteristic $\mathrm{char}(k) \neq 2$. Let $A = k[x]$ and let

$$B = k[x, y]/(2y^2 - 3x^3 + x).$$

View A as the sub-k-algebra of B generated by x.

(a) Show that B is a free A-module.

(b) Show that B is integral over A.

(c) Let $\mathfrak{p}_1 = (x - 1) \subset A$. Find a prime ideal $\mathfrak{q}_1$ of B such that $\mathfrak{q}_1 \cap A = \mathfrak{p}_1$.

(d) Let $\mathfrak{p}_2 = (0) \subset A$. Does there exist a prime ideal $\mathfrak{q}_2$ of B contained in $\mathfrak{q}_1$ such that $\mathfrak{q}_2 \cap A = \mathfrak{p}_2$?

There are **5** questions in total. The exam continues on the back of this sheet.

(4) Let R be a ring and $r \in R$, and set $S = \{1, r, r^2, \ldots\}$.
  (a) Construct an R-algebra isomorphism
$$R[x]/(rx - 1) \to S^{-1}R.$$

Now suppose that R is a discrete valuation ring (not a field), and that $r \in R$ has valuation 1. Let $A = R[x]$ and let $I = (rx - 1)$, an ideal of A.
  (b) What is the dimension of A?
  (c) Show that I is maximal.
  (d) What is the dimension of A/I?
  (e) What is the dimension of the local ring A_I?

(5) Let k be a field, s a positive integer, and $d_1, \ldots, d_s$ be positive integers. Let $R = k[x_1, \ldots, x_s]$. Given any $n \in \mathbb{Z}_{\geq 0}$ define R_n to be the k-vector subspace of R spanned by monomials $\prod_{i=1}^{s} x_i^{e_i}$ with $\sum_i e_i d_i = n$.
  (a) Show that $(R, (R_n)_n)$ is a graded ring.
  (b) Show that the Poincaré series $P(R, t)$ is given by $\prod_{i=1}^{s}(1 - t^{d_i})^{-1}$, where we take the additive function λ to be the dimension as k-vector space.
  (c) Is it true for every choice of s and the d_i that the function
$$\mathbb{N} \to \mathbb{Z}; \qquad n \mapsto \dim_k R_n$$
is polynomial in n for sufficiently large n? Give a proof or counterexample.
  (d) For a positive integer m, define $G_m = \bigoplus_{n \geq m} R_n$. Then the G_m form a decreasing sequence of subgroups of R, hence they induce the structure of a topological group on R (in which the G_m form a fundamental system of neighbourhoods of 0). Show that this topology is Hausdorff.